UNCLASSIFIED

AD 404 918

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-3-5

CATALOGED BY ASTIA
AS AD No. 404918

13 April 1963

RSIC-10

ON THE PLEONASMS IN THE GENERAL SOLUTION OF EQUILIBRIUM EQUATIONS OF AN ANISOTROPIC BODY IN DISPLACEMENT

By

E. N. Baida

From

Izvestiya Vysskikh Uchebnoikh, Zavedenii
Stro-vo I Arkhitekt No. 1, 27-37 (1959)

404 918

Redstone Scientific Information Center

U S ARMY MISSILE COMMAND
REDSTONE ARSENAL, ALABAMA

DDC
RECEIVED
MAY 2 0 1963
TISIA A

RSA FORM 1385, 1 JAN 63 PREVIOUS EDITION IS OBSOLETE

Destroy; do not return.

DDC Availability Notice. Qualified requestors may obtain copies of this report from DDC.

Publicly available from: Office of Technical Services
Department of Commerce
Washington 25, D. C.

13 April 1963 RSIC-10

ON THE PLEONASMS IN THE GENERAL SOLUTION OF EQUILIBRIUM EQUATIONS OF AN ANISOTROPIC BODY IN DISPLACEMENT

By

Balda, E. N.

From
Izvestiya Vysskikh Uchebnoikh, Zavedenii
Stro-vo I Arkhitekt No. 1, 27-37 (1959)

Translated From The Russian By
Ingeborg V. Baker

Translation Branch
Redstone Scientific Information Center
Directorate of Research and Development
Army Missile Command
Redstone Arsenal, Alabama

ON THE PLEONASMS IN THE GENERAL SOLUTION OF EQUILIBRIUM EQUATIONS OF AN ANISOTROPIC BODY IN DISPLACEMENT

By

E. N. Baida

If the system of differential equilibrium equations in displacements of an anisotropic body during the absence of volumetric forces is written in the form of an operator:

$$\left.\begin{aligned} D_1 u + D_2 v + D_3 w = 0, \\ D_4 u + D_5 v + D_6 w = 0, \\ D_7 u + D_8 v + D_9 w = 0; \end{aligned}\right\} \tag{1}$$

then it is easy to find a solution to this system in the form of:

$$\left.\begin{aligned} u &= \begin{vmatrix} D_5 D_6 \\ D_8 D_9 \end{vmatrix} \Phi_1 + \begin{vmatrix} D_3 D_2 \\ D_9 D_8 \end{vmatrix} \Phi_2 + \begin{vmatrix} D_2 D_3 \\ D_5 D_6 \end{vmatrix} \Phi_3, \\ v &= \begin{vmatrix} D_6 D_4 \\ D_9 D_7 \end{vmatrix} \Phi_1 + \begin{vmatrix} D_1 D_3 \\ D_7 D_9 \end{vmatrix} \Phi_2 + \begin{vmatrix} D_3 D_1 \\ D_6 D_4 \end{vmatrix} \Phi_3, \\ w &= \begin{vmatrix} D_4 D_5 \\ D_7 D_8 \end{vmatrix} \Phi_1 + \begin{vmatrix} D_2 D_1 \\ D_8 D_7 \end{vmatrix} \Phi_2 + \begin{vmatrix} D_1 D_2 \\ D_4 D_5 \end{vmatrix} \Phi_3, \end{aligned}\right\} \tag{2}$$

where

$$\begin{vmatrix} D_1 D_2 D_3 \\ D_4 D_5 D_6 \\ D_7 D_8 D_9 \end{vmatrix} \Phi_{i=1,2,3} = 0. \tag{3}$$

The solutions (2) and (3) may be considered as being analogous to the Bussinesk - Galerkin solution.

The presence of pleonasms in Bussinesk - Galerkin solutions were noticed for the first time by P. F. Papkovich. A more detailed examination of the questions dealing with pleonasms in the Bussinesk-Galerkin solutions is listed in the work of A. S. Maliev[2].

Applicable to the solution of (2) and (3) a certain result is given below which enables us to find all those elements of functions $\Phi_{i=1,2,3}$, the calculation of which does not alter the displacement vector. Based on this result it is possible, for example, to reduce the number of the utilized functions $\Phi_{i=1,2,3}$ to one for solving equations (2) and (3).

Let us begin by substantiating the result. For this purpose we initially transform expressions (1), (2), (3).

The system of equations (1) is rewritten with other symbols of differential operators and obtains:

$$\left.\begin{aligned} D_{x,1}u + D_{y,1}v + D_{z,1}w = 0, \\ D_{x,2}u + D_{y,2}v + D_{z,2}w = 0, \\ D_{x,3}u + D_{y,3}v + D_{z,3}w = 0. \end{aligned}\right\} \qquad (4)$$

In these equations the linear differential operators with constant coefficients $D_{x,i=1,2,3}$, $D_{y,j=1,2,3}$, $D_{z,k=1,2,3}$ appear as:

$$D_{x,1}(\ \ ) = A_{11}\frac{\partial^2}{\partial x^2} + A_{66}\frac{\partial^2}{\partial y^2} + A_{55}\frac{\partial^2}{\partial z^2} + 2A_{56}\frac{\partial^2}{\partial y \partial z} + 2A_{16}\frac{\partial^2}{\partial x \partial y} + 2A_{15}\frac{\partial^2}{\partial x \partial z};$$

$$D_{y,1}(\ \ ) = A_{16}\frac{\partial^2}{\partial x^2} + A_{26}\frac{\partial^2}{\partial y^2} + A_{45}\frac{\partial^2}{\partial z^2} +$$

$$+ (A_{25} + A_{46})\frac{\partial^2}{\partial y \partial z} + (A_{12} + A_{66})\frac{\partial^2}{\partial x \partial y} + (A_{14} + A_{56})\frac{\partial^2}{\partial x \partial z};$$

$$D_{z,1}(\ \ ) = A_{15}\frac{\partial^2}{\partial x^2} + A_{46}\frac{\partial^2}{\partial y^2} + A_{35}\frac{\partial^2}{\partial z^2} +$$

$$+ (A_{36} + A_{45})\frac{\partial^2}{\partial y \partial z} + (A_{14} + A_{56})\frac{\partial^2}{\partial x \partial y} + (A_{13} + A_{55})\frac{\partial^2}{\partial x \partial z};$$

$$D_{x,2}(\ \ ) = A_{16}\frac{\partial^2}{\partial x^2} + A_{26}\frac{\partial^2}{\partial y^2} + A_{45}\frac{\partial^2}{\partial z^2} +$$

$$+ (A_{25} + A_{46})\frac{\partial^2}{\partial y \partial z} + (A_{66} + A_{12})\frac{\partial^2}{\partial x \partial y} + (A_{56} + A_{14})\frac{\partial^2}{\partial x \partial z};$$

$$D_{y,2}(\ \ ) = A_{66}\frac{\partial^2}{\partial x^2} + A_{22}\frac{\partial^2}{\partial y^2} + A_{44}\frac{\partial^2}{\partial z^2} + 2A_{24}\frac{\partial^2}{\partial y \partial z} + 2A_{26}\frac{\partial^2}{\partial x \partial y} + 2A_{46}\frac{\partial^2}{\partial x \partial z};$$

$$D_{z,2}(\ \ ) = A_{56}\frac{\partial^2}{\partial x^2} + A_{24}\frac{\partial^2}{\partial y^2} + A_{34}\frac{\partial^2}{\partial z^2} +$$

$$+ (A_{23} + A_{44})\frac{\partial^2}{\partial y \partial z} + (A_{25} + A_{46})\frac{\partial^2}{\partial x \partial y} + (A_{36} + A_{45})\frac{\partial^2}{\partial x \partial z};$$

$$D_{x,3}(\ \ ) = A_{15}\frac{\partial^2}{\partial x^2} + A_{46}\frac{\partial^2}{\partial y^2} + A_{35}\frac{\partial^2}{\partial z^2} +$$

$$+ (A_{45} + A_{36})\frac{\partial^2}{\partial y \partial z} + (A_{14} + A_{56})\frac{\partial^2}{\partial x \partial y} + (A_{13} + A_{55})\frac{\partial^2}{\partial x \partial z};$$

$$D_{y,3}(\ \ ) = A_{56}\frac{\partial^2}{\partial x^2} + A_{24}\frac{\partial^2}{\partial y^2} + A_{34}\frac{\partial^2}{\partial z^2} +$$

$$+ (A_{44} + A_{23})\frac{\partial^2}{\partial y \partial z} + (A_{25} + A_{46})\frac{\partial^2}{\partial x \partial y} + (A_{45} + A_{36})\frac{\partial^2}{\partial x \partial z};$$

$$D_{z,3}(\ \ ) = A_{55}\frac{\partial^2}{\partial x^2} + A_{44}\frac{\partial^2}{\partial y^2} + A_{33}\frac{\partial^2}{\partial z^2} +$$

$$+ 2A_{34}\frac{\partial^2}{\partial y \partial z} + 2A_{15}\frac{\partial^2}{\partial x \partial y} + 2A_{35}\frac{\partial^2}{\partial x \partial z};$$

By inserting the significance of vector projections into the indicated operators, the equations system (4) will be presented in this manner:

$$\left.\begin{aligned} \bar{D}_x \cdot \bar{V} = 0, \\ \bar{D}_y \cdot \bar{V} = 0, \\ \bar{D}_z \cdot \bar{V} = 0, \end{aligned}\right\} \qquad (5)$$

in which

$$\left.\begin{aligned} \bar{D}_x = \bar{i} \cdot D_{x,1} + \bar{j} \cdot D_{y,1} + \bar{k} D_{z,1}, \\ \bar{D}_y = \bar{i} \cdot D_{x,2} + \bar{j} D_{y,2} + \bar{k} D_{z,2}, \\ \bar{D}_z = \bar{i} D_{x,3} + \bar{j} D_{y,3} + \bar{k} D_{z,3} \end{aligned}\right\} \qquad (6)$$

are indicated.

If the left parts of each equation of the system (5) are correspondingly multiplied by $\bar{i}, \bar{j}, \bar{k}$, and the results summated, then the transcript of the equation system (5) can be written in the form of a parity vector

$$\bar{i} \cdot (\bar{D}_x \cdot \bar{V}) + \bar{j} \cdot (\bar{D}_y \cdot \bar{V}) + \bar{k} \cdot (\bar{D}_z \cdot \bar{V}) = 0. \qquad (7)$$

Taking into consideration the new symbols, we rewrite the solution of (2) and (3) as:

$$\bar{V}=(\bar{D}_y\times\bar{D}_z)\Phi_x+(\bar{D}_z\times\bar{D}_x)\Phi_y+(\bar{D}_x\times\bar{D}_y)\bar{\Phi}_z, \tag{8}$$

$$[\bar{D}_x\cdot(\bar{D}_y\times\bar{D}_z)]\bar{\Phi}=0, \tag{9}$$

where

$$\bar{\Phi}=\bar{i}\Phi_x+\bar{j}\Phi_y+\bar{k}\Phi_z. \tag{10}$$

Symbols $\Phi_{i=1,2,3}$ in expressions (8) and (10) are accordingly changed to $\Phi_{x,y,z}$.

Considering that

$$\begin{aligned}
&\bar{D}_x\cdot(\bar{D}_z\times\bar{D}_x)\equiv 0, \qquad &&\bar{D}_y\cdot(\bar{D}_x\times\bar{D}_y)\equiv 0,\\
&\bar{D}_x\cdot(\bar{D}_x\times\bar{D}_y)\equiv 0, &&\bar{D}_z\cdot(\bar{D}_y\times\bar{D}_z)\equiv 0,\\
&\bar{D}_y\cdot(\bar{D}_y\times\bar{D}_z)\equiv 0, &&\bar{D}_z\cdot(\bar{D}_z\times\bar{D}_x)\equiv 0
\end{aligned}$$

and using the possibility of circular permutation for a mixed product of vector - operators, it is easy to verify the solution of the equation (7) by (8) and (9).

Before we begin to analyze the elements superfluous from the viewpoint of general solutions (8) and (9), let us investigate the differential operators with constant coefficients appearing as:

$$D_1(\ )=\sum_{i+j+k=m}A_{ijk}\frac{\partial^m}{\partial x^i\partial y^j\partial z^k}, \tag{11}$$

$$D_2(\ )=\sum_{i_1+j_1+k_1=l}B_{i_1j_1k_1}\frac{\partial^l}{\partial x^{i_1}\partial y^{j_1}\partial z^{k_1}}.$$

Let us assume for determination that all A_{ijk} and $B_{i_1j_1k_1}\neq 0$.

Let us prove that, with certain limitations imposed on operators $D_1(\ )$ and $D_2(\ )$, a partial solution of the equation exists:

$$D_1 f = \varphi \tag{12}$$

in the form $f = f_0$, where

$$D_2 f_0 = 0, \tag{13}$$

if φ is any given function from

$$D_2 \varphi = 0. \tag{14}$$

Let us seek the solution in a series of functions presented by means of a finite sum of homogeneous polynomials. Assuming

$$f_0 = \sum_{s_1=0}^{n+m} f_{s_1}$$

and

$$\varphi = \sum_{s=0}^{n} \varphi_s.$$

Here $n \gg m, l$.

Functions φ_s and f_s, are presented by means of the homogeneous polynomials in series s and s_1, in this manner:

$$\varphi_s = \sum_{\nu+\varkappa+\eta=s} a'_{\nu\varkappa\eta} x^\nu y^\varkappa z^\eta \tag{15}$$

and

$$f_{s_1} = \sum_{\nu_1+\varkappa_1+\eta_1=s_1} b'_{\nu_1\varkappa_1\eta_1} x^{\nu_1} y^{\varkappa_1} z^{\eta_1}. \tag{16}$$

It is necessary that φ_s and f_s should satisfy $D_1 f_{s1} = \varphi_s$ and $D_2(\ ) = 0$.

Let us take $s_1 = s + m$.

We substitute (15) and (16) into (12) and consider (11).

Then
$$\sum_{\nu_1+\varkappa_1+\eta_1=s+m}\ \sum_{l+j+k=m} A_{ljk}\nu_1(\nu_1-1)\dots(\nu_1-l+1)\eta_1(\eta_1-1)\dots(\eta_1-k+1)\times$$
$$\times\varkappa_1(\varkappa_1-1)\dots(\varkappa_1-j+1)\cdot b'_{\nu_1\varkappa_1\eta_1}x^{\nu_1-l}y^{\varkappa_1-j}z^{\eta_1-k}=$$
$$=\sum_{\nu+\varkappa+\eta=s} a'_{\nu\varkappa\eta}x^{\nu}y^{\varkappa}z^{\eta}. \qquad (17)$$

The left and right sides of parity (17) represent homogeneous polynomials of series s. It can easily be seen, that each $x^{\nu}y^{\varkappa}z^{\eta}$ on the right side of parity (17) corresponds to such $x^{\nu_1-l}y^{\varkappa_1-j}z^{\eta_1-k}$ on the left side, in which

$$\left.\begin{aligned}\nu_1-l&=\nu,\\ \varkappa_1-j&=\varkappa,\\ \eta_1-k&=\eta.\end{aligned}\right\} \qquad (18)$$

Considering (18), parity (17) will be rewritten differently:

$$\sum_{\nu+\varkappa+\eta=s}\ \sum_{l+j+k=m} A_{ljk}(\nu+l)\dots(\nu+1)(\varkappa+j)\dots(\varkappa+1)(\eta+k)\dots(\eta+1)\times$$
$$\times b'_{(\nu+l),(\varkappa+j),(\eta+k)}\cdot x^{\nu}y^{\varkappa}z^{\eta}=\sum_{\nu+\varkappa+\eta=s} a'_{\nu\varkappa\eta}x^{\nu}y^{\varkappa}z^{\eta}. \qquad (19)$$

Parity (19) may be satisfied by accepting:

$$\sum_{l+j+k=m} A_{ljk}(\nu+l)\dots(\nu+1)(\varkappa+j)\dots(\varkappa+1)(\eta+k)\dots(\eta+1)\times$$
$$\times b'_{(\nu+l),(\varkappa+j),(\eta+k)}=a'_{\nu\varkappa\eta}. \qquad (20)$$

Here $\nu+\chi+\eta=s$.

Let us assume that $b'_{(\nu+l),(\varkappa+j),(\eta+k)}=\dfrac{b_{(\nu+l),(\varkappa+j),(\eta+k)}}{(\nu+l)!\ (\varkappa+j)!(\eta+k)!}$ (21)

and

$$a'_{\nu\varkappa\eta}=\frac{a_{\nu\varkappa\eta}}{\nu!\varkappa!\eta!}. \qquad (22)$$

Taking into consideration (21) and (22), parity (20) is rewritten as:

$$\sum_{l+j+k=m} A_{ljk}b_{(\nu+l),(\varkappa+j),(\eta+k)}=a_{\nu\varkappa\eta}, \qquad (23)$$

where $\nu+\chi+\eta=s$.

Analogously from (13) and (14) it follows, that:

$$\sum_{l_1+j_1+k_1=l} B_{l_1 j_1 k_1} b_{(\gamma_2+l_1),(\chi_2+j_1),(\eta_2+k_1)} = 0, \qquad (24)$$

where $\gamma_2 + \chi_2 + \eta_2 = s - l + m$

and
$$\sum_{l_1+j_1+k_1=l} B_{l_1 j_1 k_1} a_{(\gamma_3+l_1),(\chi_3+j_1),(\eta_3+k_1)} = 0, \qquad (25)$$

where $\gamma_3 + \chi_3 + \eta_3 = s - l$.

We will seek a solution of the system of finite-difference equations (23) and (24). Since function φ is defined, then all $\alpha_{\gamma\chi\eta}$ can be considered as known.

Let us express the coefficients $\alpha_{\gamma\chi\eta}$ as follows:

$$a_{\gamma\chi\eta} = \frac{\partial^s \varphi_s}{\partial x^\gamma \partial y^\chi \partial z^\eta}. \qquad (26)$$

By means of

$$B_{l00} \cdot \frac{\partial^l \varphi_s}{\partial x^l} = - \sum_{\substack{l_1+j_1+k_1=l \\ l_1 \neq l}} B_{l_1 j_1 k_1} \frac{\partial^l \varphi_s}{\partial x^{l_1} \partial y^{j_1} \partial z^{k_1}}$$

t is possible in (26) to exclude the differentiation in respect to x of an order higher than 1 .

In this manner, only those coefficients can be independent from a number of $a_{\gamma\chi\eta}$, in which $\gamma = 0, 1, 2 \ldots (l-1)$. In essence, these coefficients are:

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
a_{00s} \\
a_{01(s-1)} \\
a_{02(s-2)} \\
\dots\dots\dots\dots
\end{array}\right\} s+1, \\
\left.\begin{array}{l}
a_{10(s-1)} \\
a_{11(s-2)} \\
a_{12(s-3)} \\
\dots\dots\dots\dots
\end{array}\right\} s, \\
\left.\begin{array}{l}
\dots\dots\dots\dots \\
a_{(l-1)0(s-l+1)} \\
a_{(l-1)1(s-l)} \\
a_{(l-1)2(s-l-1)} \\
\dots\dots\dots\dots
\end{array}\right\} s-l+2.
\end{array}\right|
\begin{array}{l}
\text{Their general number equals} \\
\dfrac{(2s-l+3)}{2}\cdot l.
\end{array}
$$

The independent coefficients $\alpha_{\gamma x\eta}$ are presented as follows:

$$
\left.\begin{array}{l}
a_{0x\eta} = C_1^{(0)}\beta_1^x\gamma_1^\eta + C_2^{(0)}\beta_2^x\gamma_2^\eta + \dots + C_{s+1}^{(0)}\beta_{s+1}^x\gamma_{s+1}^\eta, \\
\text{where } x+\eta = s; \\
a_{1x\eta} = C_1^{(1)}\beta_1^x\gamma_1^\eta + C_2^{(1)}\beta_2^x\gamma_2^\eta + \dots + C_s^{(1)}\beta_s^x\gamma_s^\eta, \\
\text{where } x+\eta = s-1 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
a_{(l-1)x\eta} = C_1^{(l-1)}\beta_1^x\gamma_1^\eta + C_2^{(l-1)}\beta_2^x\gamma_2^\eta + \dots + C_{s-l+2}^{(l-1)}\beta_{s-l+2}^x\gamma_{s-l+2}^\eta, \\
\text{where } x+\eta = s-l+1.
\end{array}\right\} \quad (27)
$$

Here

$$
\left.\begin{array}{l}
l\left\{\begin{array}{l}
C_1^{(0)} = C_{1,1} + C_{1,2} + \dots + C_{1,l}, \\
C_1^{(1)} = C_{1,1}\cdot\alpha_{1,1} + C_{1,2}\cdot\alpha_{1,2} + \dots + C_{1,l}\cdot\alpha_{1,l}, \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
C_1^{(l-1)} = C_{1,1}\cdot\alpha_{1,1}^{l-1} + C_{1,2}\cdot\alpha_{1,2}^{l-1} + \dots + C_{1,l}\cdot\alpha_{1,l}^{l-1},
\end{array}\right. \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
l\left\{\begin{array}{l}
C_p^{(0)} = C_{p,1} + C_{p,2} + \dots + C_{p,l}, \\
C_p^{(1)} = C_{p,1}\cdot\alpha_{p,1} + C_{p,2}\cdot\alpha_{p,2} + \dots + C_{p,l}\cdot\alpha_{p,l}, \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
C_p^{(l-1)} = C_{p,1}\cdot\alpha_{p,1}^{l-1} + C_{p,2}\cdot\alpha_{p,2}^{l-1} + \dots + C_{p,l}\cdot\alpha_{p,l}^{l-1};
\end{array}\right. \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}\right\} \quad (28)
$$

where $p < 1, 2 \ldots (s - l + 2)$.

Analogously with $p > s - l + 2$ it is necessary to adopt

$$\left.\begin{array}{l} l-1 \left\{\begin{array}{l} C^{(0)}_{s-l+3} = C_{(s-l+3),1} + C_{(s-l+3),2} + \ldots + C_{(s-l+3),l}, \\ C^{(1)}_{s-l+3} = C_{(s-l+3),1} \cdot \alpha_{(s-l+3),1} + C_{(s-l+3),2} \times \\ \times \alpha_{(s-l+3),2} + \ldots + C_{(s-l+3),l} \cdot \alpha_{(s-l+3),l}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C^{(l-2)}_{s-l+3} = C_{(s-l+3),1} \cdot \alpha^{l-2}_{(s-l+3),1} + C_{(s-l+3),2} \cdot \alpha^{l-2}_{(s-l+3),2} + \ldots + \\ + C_{(s-l+3),l} \cdot \alpha^{l-2}_{(s-l+3),l}; \end{array}\right. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 2 \left\{\begin{array}{l} C^{(0)}_{s} = C_{s,1} + C_{s,2} + \ldots + C_{s,l}, \\ C^{(1)}_{s} = C_{s,1} \cdot \alpha_{s,1} + C_{s,2} \cdot \alpha_{s,2} + \ldots + C_{s,l} \cdot \alpha_{s,l}; \end{array}\right. \\ C^{(0)}_{s+1} = C_{(s+1),1} + C_{(s+1),2} + \ldots + C_{(s+1),l}. \end{array}\right\} \quad (29)$$

Expressions

$$l \left\{\begin{array}{l} \alpha_{p,1}, \beta_p, \gamma_p; \\ \alpha_{p,2}, \beta_p, \gamma_p; \\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot \\ \alpha_{p,l}, \beta_p, \gamma_p \end{array}\right\} \quad (30)$$

represent 1 solution of the characteristic equation of operator $D_2(\ )$. Here β_p and γ_p are so selected that among the ratios

$$\left.\frac{\beta_p}{\gamma_p}\right|_{p=1,2,3\ldots(s+1)} \quad (31)$$

none are alike. The below assumed limitation concerning absence of multiple roots among $\alpha_{p,1}; \alpha_{p,2}; \ldots \alpha_{p,l}$ is not compulsory and can be removed by altering the form of expressions (28), (29). If, for example, during concrete value P it is found that $\alpha_{p,1} = \alpha_{p,2}$, then in (28), (29), instead of $C_{p,2}, C_{p,2} \cdot \alpha_{p,2}, C_{p,2} \cdot \alpha^2_{p,2}, \ldots C_{p,2} \cdot \alpha^{l-1}_{p,2}$ the following should be respectively taken $0, C_{p,2} \cdot \alpha_{p,1}, C_{p,2} \cdot 2\alpha^2_{p,1}, \ldots, C_{p,2} \cdot (l-1)\, \alpha^{l-1}_{p,1}$.

The unknown coeffieient $C^{(q)}_p$, ,

where $$p = 1,2 \ldots (s+1),$$ $$q = 0,1 \ldots (l-1).$$

can be determined by means of solving of ℓ algebraic systems (27), the consistency of which is easily established by the fact that the determinants are irreversible to zero:

$$\Delta_q = \begin{vmatrix} \beta_1^0\gamma_1^{s-q} & \beta_2^0\gamma_2^{s-q} & \dots & \beta_{s+1-q}^0\gamma_{s+1-q}^{s-q} \\ \beta_1^1\gamma_1^{s-q-1} & \beta_2^1\gamma_2^{s-q-1} & \dots & \beta_{s+1-q}^1\gamma_{s+1-q}^{s-q-1} \\ \beta_1^2\gamma_1^{s-q-2} & \beta_2^2\gamma_2^{s-q-2} & \dots & \beta_{s+1-q}^2\gamma_{s+1-q}^{s-q-2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \beta_1^{s-q}\gamma_1^0 & \beta_2^{s-q}\gamma_2^0 & \dots & \beta_{s+1-q}^{s-q}\gamma_{s+1-q}^0 \end{vmatrix}, \quad q=0,1\dots(l-1) \qquad (32)$$

Actually, by presenting (32) in the form of

$$\Delta_q = \gamma_1^{s-q}\cdot\gamma_2^{s-q}\cdots\gamma_{s+1-q}^{s-q} \begin{vmatrix} \left(\frac{\beta_1}{\gamma_1}\right) & \left(\frac{\beta_2}{\gamma_2}\right) & \dots & \left(\frac{\beta_{s+1-q}}{\gamma_{s+1-q}}\right) \\ \left(\frac{\beta_1}{\gamma_1}\right)^2 & \left(\frac{\beta_2}{\gamma_2}\right)^2 & \dots & \left(\frac{\beta_{s+1-q}}{\gamma_{s+1-q}}\right)^2 \\ \dots & \dots & \dots & \dots \\ \left(\frac{\beta_1}{\gamma_1}\right)^{s-q} & \left(\frac{\beta_2}{\gamma_2}\right)^{s-q} & \dots & \left(\frac{\beta_{s+1-q}}{\gamma_{s+1-q}}\right)^{s-q} \end{vmatrix}, \quad q=0,1,2\dots(l-1) \qquad (33)$$

and considering that the determinants entering into (33) are Vandermonde type determinants, allowing Δ_q to be expressed in the form:

$$\Delta_q = \left[\prod_{p=1,2\dots(s+1-q)}\gamma_p^{s-q}\right]\left[\prod_{\substack{p_1>p_2\\ p_1,p_2=1,2\dots(s+1-q)}}\left(\frac{\beta_{p_1}}{\gamma_{p_1}}-\frac{\beta_{p_2}}{\gamma_{p_2}}\right)\right],$$

we note that, with realization of (31), $\Delta_q \neq 0$.

The existence of solutions of the algebraic systems of equations (28) relative to $C_{p,r}$ is analogously proven:

where $$p = 1, 2\dots(s-l+2),\quad r = 1, 2\dots l$$

and of systems (29) relative to $C_{p,r}$,

where
$$p=(s-l+3),\ (s-l+4)\dots(s+1),$$
$$r=1,2\dots l.$$

Morever, for algebraic systems (29) the solutions are not univalent, due to incompleteness of systems (29). The consistency of systems (28) and (29) is evident.

Now, the given coefficients $\alpha_{\gamma\chi\eta}$ are presented in this form:

$$a_{\nu\chi\eta}=\sum_{p=1,2\dots(s+1)}\ \sum_{r=1,2\dots l} C_{p,r}\alpha_{p,r}^{\nu}\beta_{p}^{\chi}\gamma_{p}^{\eta}, \tag{34}$$

where $\nu+\chi+\eta=s$.

Here $C_{p,r}\Big|_{\substack{r=1,2\dots l\\ p=1,2\dots(s+1),}}$ are determined from the correlations

(27), (28), (29). It is easily seen that (34) does not contradict (27), (28), (29), and consequently (30) solves the finite-difference equation (25).

Let's seek solutions of (23), (24) in the form:

$$b_{\nu_1\chi_1\eta_1}=\sum_{p=1,2\dots(s+1)}\ \sum_{r=1,2\dots l} E_{p,r}\cdot\alpha_{p,r}^{\nu_1}\beta_{p}^{\chi_1}\gamma_{p}^{\eta_1}, \tag{35}$$

where $\nu_1+\chi_1+\eta_1=s_1$.

Selection of $b_{\nu_1\chi_1\eta_1}$ in form (35) allows satisfaction of condition (24) with $E_{p,r}$.

Let's substitute (35) into (23), considering (18). Then

$$\sum_{i+j+k=m} A_{ijk}\Big(\sum_{p=1,2\dots(s+1)}\ \sum_{r=1,2\dots l} E_{p,r}\cdot\alpha_{p,r}^{\nu+i}\beta_{p}^{\chi+j}\gamma_{p}^{\eta+k}\Big)= \\ =\sum_{p=1,2\dots(s+1)}\ \sum_{r=1,2\dots l} C_{p,r}\cdot\alpha_{p,r}^{\nu}\beta_{p}^{\chi}\gamma_{p}^{\eta}. \tag{36}$$

Transforming (36) in the form of:

$$\sum_{p=1,2\,..\,(s+1)} \; \sum_{r=1,2\,..\,l} \; \sum_{l+j+k=m} A_{ljk} E_{p,r} \cdot \alpha_{p,r}^{\nu+l} \beta_p^{\varkappa+j} \gamma_p^{\eta+k} = \\ = \sum_{p=1,2\,..\,(s+1)} \; \sum_{r=1,2\,..\,l} C_{p,r} \cdot \alpha_{p,r}^{\nu} \beta_p^{\varkappa} \gamma_p^{\eta}. \tag{37}$$

In the parity (37) we assume:

$$\sum_{l+j+k=m} A_{ljk} E_{p,r} \cdot \alpha_{p,r}^{\nu+l} \beta_p^{\varkappa+j} \gamma_p^{\eta+k} = C_{p,r} \cdot \alpha_{p,r}^{\nu} \beta_p^{\varkappa} \gamma_p^{\eta}$$

or

$$E_{p,r} \cdot \alpha_{p,r}^{\nu} \beta_p^{\varkappa} \gamma_p^{\eta} \sum_{l+j+k=m} A_{ljk} \cdot \alpha_{p,r}^{l} \beta_p^{j} \gamma_p^{k} = C_{p,r} \cdot \alpha_{p,r}^{\nu} \beta_p^{\varkappa} \gamma_p^{\eta}. \tag{38}$$

Solving (38) relative to $E_{p,r}$, we get:

$$E_{p,r} = \frac{C_{p,r}}{\sum\limits_{l+j+k=m} A_{ljk}\, \alpha_{p,r}^{l} \beta_p^{j} \gamma_p^{k}},$$

where

$$p = 1, 2 \ldots (s+1), \\ r = 1, 2 \ldots l.$$

Let's restrict operators $D_1(\;)$ and $D_2(\;)$ to the condition [Q] of possibility of such selection for each p $\alpha_{p,r}, \beta_p, \gamma_p \big|_{r=1,2\ldots l}$ by compliance with (31), during which

$$\sum_{l+j+k=m} A_{ljk} \cdot \alpha_{p,r}^{l} \beta_p^{j} \gamma_p^{k} \neq 0, \tag{39}$$

where $r = 1,2 \ldots l$;

$$\sum_{l_1+j_1+k_1=l} B_{l_1 j_1 k_1} \cdot \alpha_{p,r}^{l_1} \beta_p^{j_1} \gamma_p^{k_1} = 0,$$

where $r = 1,2 \ldots l$.

The left sides of the written expressions are the characteristic forms of the operators $D_1(\;)$ and $D_2(\;)$.

Thus, coefficients $b_{\nu_1\varkappa_1\eta_1}$, being restricted on account of (39), are easily determined, satisfying all set conditions; that is, expressions (23) and (24). $b_{\nu_1\varkappa_1\eta_1}$ remain the undetermined coefficients for which $\gamma_4+\chi_4+\eta_4<m$. These coefficients can be considered as arbitrary.

We write down the final solution of the given problem:

$$f_0=\sum_{s_1=0}^{n+m} f_{s_1},$$

where

$$f_{s_1}=\sum_{\nu_1+\varkappa_1+\eta_1=s_1}\frac{b_{\nu_1\varkappa_1\eta_1}}{\nu_1!\,\varkappa_1!\,\eta_1!}x^{\nu_1}y^{\varkappa_1}z^{\eta_1}.$$

It is easy to see that, for the realization of condition [Q], it is necessary that the characteristic forms of the operators D_1 ( ) and D_2 ( ) should not be decomposed by the factors, different from the constant, among which is the common factor.

Lets return to the examination of the basic question, concerning the pleonasms in the general solution (8), (9), indicated at the beginning of this chapter. From equation

$$\left(\overline{D}_y\times\overline{D}_z\right)\psi_x+\left(\overline{D}_z\times\overline{D}_x\right)\psi_y+\left(\overline{D}_x\times\overline{D}_y\right)\psi_z=0 \qquad (40)$$

we will try to determine the components of a certain vector $\overline{\psi}$.

It is possible to represent the vector of the volumetric force $\overline{F}$ in the form of

$$\overline{F}=-\overline{i}\left(\overline{D}_x\cdot\overline{V}\right)-\overline{j}\left(\overline{D}_y\cdot\overline{V}\right)-\overline{k}\left(\overline{D}_z\cdot\overline{V}\right)$$

which allows finding a general solution to (40) in the form

$$\bar{\psi} = \bar{i}\left(\bar{D}_x \cdot \bar{\theta}\right) + \bar{j}\left(\bar{D}_y \cdot \bar{\theta}\right) + \bar{k}\left(\bar{D}_z \cdot \bar{\theta}\right). \tag{41}$$

Substituting (41) into (40) and performing a transformation, we should finally obtain an equation which would be satisfied by the components of vector θ.

This equation is:

$$\left[\bar{D}_x \cdot \left(\bar{D}_y \times \bar{D}_z\right)\right](\theta_x + \theta_y + \theta_z) = 0, \tag{42}$$

The vector $\bar{\psi}$, obtained in this manner, as can be easily seen, may be added to vector $\bar{\Phi}$ in solving (8) without altering the displacement vector $\bar{V}$.

By this, the vector $(\bar{\Phi} + \bar{\psi})$, as vector $\bar{\psi}$, in accordance with (9), (41), (42), does not necessarily have to satisfy the correlation:

$$\left[\bar{D}_x \cdot \left(\bar{D}_y \times \bar{D}_z\right)\right](\quad) = 0. \tag{43}$$

We will show that, without losing generality in solving (8), (9), it is sufficient to retain one function, for example, Φ_x.

From equation (42) it follows, that

$$\theta_x = -\theta_y - \theta_z + \varphi, \tag{44}$$

where φ is any function, satisfying expression (43).

In parity (44) it may be presumed that θ_y and θ_z are the arbitrary functions of their arguments x, y, z.

Considering that

$$\bar{V} = \left(\bar{D}_y \times \bar{D}_z\right)\Phi_x + \left(\bar{D}_z \times \bar{D}_x\right)\Phi_y + \left(\bar{D}_x \times \bar{D}_y\right)\Phi_z =$$
$$= \left(\bar{D}_y \times \bar{D}_z\right)(\Phi_x + \psi_x) + \left(\bar{D}_z \times \bar{D}_x\right)(\Phi_y + \psi_y) + \left(\bar{D}_x \times \bar{D}_y\right)(\Phi_z + \psi_z),$$

we will try to dispose of functions ψ_y, ψ_z with the aid of θ_y and θ_z in such a manner that

$$\left.\begin{aligned} \psi_y &= -\Phi_y, \\ \psi_z &= -\Phi_z. \end{aligned}\right\} \tag{45}$$

In accordance with (6), (41), (44), correlation (45) is rewritten in a more explicit form as:

$$\left.\begin{aligned} \Phi_y + D_{x,2}\varphi &= (D_{x,2} - D_{y,2})\theta_y + (D_{x,2} - D_{z,2})\theta_z, \\ \Phi_z + D_{x,3}\varphi &= (D_{x,3} - D_{y,3})\theta_y + (D_{x,3} - D_{z,3})\theta_z. \end{aligned}\right\} \tag{46}$$

We introduce the designation:

$$\begin{aligned} \Phi_y + D_{x,2}\varphi &= \Phi_y', \\ \Phi_z + D_{x,3}\varphi &= \Phi_z'. \end{aligned}$$

We consolidate function φ . Then Φ_y, Φ_z may be considered as known functions.

The quotient solution (46) we will find in such a form:

$$\left.\begin{aligned} \theta_y &= (D_{x,3} - D_{z,3})\,\omega_y - (D_{x,2} - D_{z,2})\,\omega_z, \\ \theta_z &= (D_{x,2} - D_{y,2})\,\omega_z - (D_{x,3} - D_{y,3})\,\omega_y. \end{aligned}\right\} \tag{47}$$

Here w_{yz} are determined from the equation

where
$$\left.\begin{aligned} L\,\omega_{y,z} &= \Phi_{y,z}', \\ \left[\bar{D}_x \cdot \left(\bar{D}_y \times \bar{D}_z\right)\right]\Phi_{y,z}' &= 0. \end{aligned}\right\} \tag{48}$$

The operator L( ) has the following form:

$$\begin{aligned} \left(\bar{D}_y \times \bar{D}_z\right)\cdot\left(\bar{i} + \bar{j} + \bar{k}\right)(\quad) = {} & \left[(D_{x,2} - D_{y,2})(D_{x,3} - D_{z,3}) - \right. \\ & \left. - (D_{x,2} - D_{z,2})(D_{x,3} - D_{y,3})\right](\quad), \end{aligned} \tag{49}$$

If operators (43) and (49) satisfy condition [Q] , then it is

permissible to seek solution (48) in a set of functions

$$\left[\overline{D}_x \cdot \left(\overline{D}_y \times \overline{D}_z\right)\right] \omega_{y,z} = 0.$$

Taking in consideration condition $\left[D_x \cdot \left(\overline{D}_y \times \overline{D}_z\right)\right] \omega_{y,z} = 0$ and expression (47), it may be easily shown that function

$$\psi_x = D_{x,1}(-\theta_y - \theta_z + \varphi) + D_{y,1}\theta_y + D_{z,1}\theta_z \qquad (50)$$

satisfies correlation

$$\left[\overline{D}_x \cdot \left(\overline{D}_y \times \overline{D}_z\right)\right] (\quad) = 0.$$

We finally arrive at the following result:

$$\left.\begin{aligned} \overline{V} &= \left(\overline{D}_y \times \overline{D}_z\right)(\Phi_x + \psi_x), \\ \left[\overline{D}_x \cdot \left(\overline{D}_y \times \overline{D}_z\right)\right] &(\Phi_x + \psi_x) = 0. \end{aligned}\right\} \qquad (51)$$

where

In this solution, functions $\Phi_{y,z}$ are abolished by means of a special selection of $\psi_{y,z}$.

Function ψ_x, determined by formula (50), may assume different values, depending on the selection of function φ, due to which solution of (51) is not entirely free of pleonasms. If $\Phi_x + \psi_x = \Phi_x^0$ is indicated, then the solution of (51) will be finally rewritten as:

$$\overline{V} = \left(\overline{D}_y \times \overline{D}_z\right) \Phi_x^0,$$

where

$$\left[\overline{D}_x \cdot \left(\overline{D}_y \times \overline{D}_z\right)\right] \Phi_x^0 = 0.$$

Under other limitations applied to the differential operators, the remaining variants of solutions can also be obtained:

$$\overline{V} = \left(\overline{D}_z \times \overline{D}_x\right) \Phi_y^0,$$

$$\overline{V} = \left(\overline{D}_x \times \overline{D}_y\right) \Phi_z^0,$$

where Φ_y^0 and Φ_z^0 satisfy correlation (43).

REFERENCES

1. Baida, E. H., Determination of Stresses and Displacements in an Anisotropic Body by Means of Three Functions, Authors' Thesis Abstracts, publication of Leningrad Institute of Construction Engineers, 1955.

2. Maliev, A. S., On Selection of Functions in General Solutions of the Problem of Equilibrium of an Anisotropic Elastic Body, Proceedings of the Leningrad Electrotechnical Institute of Railroad Engineers, 1952.

Submitted to Editor
5 May 1958.